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The Mechanics of an Affinely-Rigid Body

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Abstract

The mechanics of an affinely-rigid body is investigated on both the classical and the quantum level. An affinely-rigid body is defined as a system of material points or a continuous medium in which all affine relations are frozen. Our treatment is based on the general theory of systems with closed teleparallelisms, presented in Section 2 of this paper.

1. *Introduction*

In Section 2 of this paper we present the general outline of the mechanics of manifolds endowed with the local structure of a group space. The theory of the affinely-rigid body presented in the following sections provides us with some special examples of such mechanics. However, having in view some important physical and geometrical applications, we develop this theory as an autonomic subject rather than as a mere example.

The usual (i.e. metrical) rigid body is a discrete or continuous system of material points, the mutual distances of which are fixed by some constraints. Hence all metrical relations between elements of such a body are frozen. An affinely-rigid body is similarly defined, such that all affine relations between its elements remain invariant during any motion. Obviously, metrical constraints are stronger than the affine ones.

In a previous paper (Sławianowski, 1974) we have given the simplified formulation of the mechanics of affinely-rigid bodies. In particular, connections with the usual notions of the theory of continuous media were investigated. From the point of view of the theory of the continuum, an affinely-rigid body is a medium the deformative behaviour of which is restricted to undergoing homogeneous deformations only.

In the present paper we give the detailed geometrical analysis of the problem in terms of the teleparallelisms.

The theory of affinely-rigid bodies is interesting from the purely geometrical point of view at least. Also, we hope it will help us to answer the following philosophical, but interesting, questions: What would happen if we got rid of the notion of a metric? In what way would we have to describe the world bereft of metric geometry?

Such problems have been formulated and studied by Bergmann & Brunnings (1949). They are not so academic as would appear; moreover, they concern the very fundamental aspects of the connection between physics and geometry. One hopes such investigations will help us to overcome some remainders of the Newtonian dualism of space-time geometry and physical phenomena which are still present in general relativity. In fact, the metric tensor (gravitation) enters into Einstein equations in a quite different way than all other physical degrees of freedom, i.e. fields and particles. Besides, it is subject to the global restrictions of non-singularity and hyperbolicity; there are no counterparts of such restrictions in 'physical' degrees of freedom. Therefore gravitation retains some features of *a priori*, absolute, Newtonian metric geometry. That was why Einstein has suspected his equations to be valid in the approximation of weak fields and small matter densities only. It seems that all efforts to overcome the dualism of gravitation and 'real' physics should start with an analysis of theories in amorphous spaces. The mechanics of affinely-rigid bodies in an affine space without metric, provides us with the simplest model of such a theory. The general ideas of such mechanics are sketched in Section 5 of this paper. An affinely-rigid body in amorphous affine space is an obvious counterpart of the usual rigid body in euclidean space.

Besides such philosophy, we have in view further practical physical applications, for example in the theory of large oscillations of molecules, small mono-crystals and atomic nuclei. Applications in the dynamics of non-primitive crystal lattices seem to be possible, especially in the case of molecular crystals; our theory could then be used to describe the additional, internal degrees of freedom of lattice points (rotations and deformations of molecules). The theory may also be useful in the statistical mechanics of polyatomic gases.

Applications in the theory of elementary particles are not excluded. In fact, kinematical and dynamical symmetries of our theory, based on the full linear group $GL(n)$, to some extent seem to be similar to the $U(n)$ symmetries. Let us notice that both groups mentioned, $GL(n)$, $U(n)$, are different real forms of the same complex Lie group $GL(n, C)$. It is possible that there is something deep and non-trivial in this fact. Besides, let us notice that the present description of internal degrees of freedom of elementary particles is based on metric geometry only—the spin operators generate (two-valued) rigid rotations around the point at which the particle is placed. Such a description is justified with good accuracy by experiments. However, from the principal point of view there is something strange and inconceivable in such 'honouring' of rigid rotations. Even when starting with the *a priori* fixed metric geometry (not the aforementioned 'amorphous' philosophy), it is hard to accept the rejection of deformative, affinely-rigid rotations. Rather, one can expect some reasonable physical corrections when the deformative (affinely-rigid) degrees of freedom are allowed ('oscillations' superposed on 'gyroscopic rotations'). Obviously, we need not conceive such deformative degrees of freedom in the classical sense of the theory of continua. Taking the 'deformative' phenomena into account can be achieved by endowing a particle with an 'affine spin' generating affine rotations around the point where the particle is placed. Let us remember that the usual (metrical) spin describes rotational degrees of freedom without suggesting the usual gyroscopic rigid rotations of extended particles.

All problems are investigated in parallel on both the classical and quantum level. Our theory is formulated in affine space; however, it is possible to formulate the theory of small ('test') affinely-rigid bodies in a manifold, in particular in a curved space-time of general relativity.

Notations

Throughout this paper the standard notation of modern books and papers on differential geometry and mechanics is consequently used. We refer mainly to Sternberg (1964), Lichnerowicz (1955), Kobayashi & Nomizu (1963), Lang (1962), Abraham (1967) and Trautman (1970). Where geometry of phase spaces is concerned we adopt the notation of earlier papers (Sławianowski, 1972, 1974); it is essentially the same as that of Śniatycki & Tulczyjew (1971) and Śniatycki (1973). Let us recall only a few symbols and fix some additional ones:

Natural projections of the tangent and cotangent bundles onto their base Q are denoted as $\tau_Q: TQ \to Q$, $\tau_Q^*: T^*Q \to Q$ respectively, or, briefly, τ, τ^*.

The canonical Pfaff form on T^*Q will be denoted as ω_Q, or, in short, ω; obviously $\omega_p = p \circ T\tau_Q^* | T_pT^*Q$. When Q is a configuration space of a system then $(T^*Q, d\omega_Q)$ is its classical phase space. A vector-field X on T^*Q, satisfying the equation $X \lrcorner d\omega_Q = -dF$, will be denoted as $d\tilde{F}$; the function F itself is referred to as its generator. Natural lifts of the vector-field $X: Q \to TQ$ to the manifolds TQ, T^*Q will be denoted as $X': TQ \to TTQ$, $\bar{X}: T^*Q \to TT^*Q$ respectively. Their local one-parameter groups are obtained by lifting the local group of X by means of the tangent functor T. Let us notice that $\bar{X}$ could be defined by the following conditions as well: (i) $T\tau_Q^* \circ \bar{X} = X \circ \tau_Q^*$ ($\bar{X}$ projects to Q onto X) and (ii) $\mathscr{L}_X\, \omega_Q = 0$. Obviously, $\bar{X}$ is a hamiltonian vector-field; its generator is a function $F_X = \langle \omega_Q, \bar{X} \rangle$. $\bar{X}$ is an infinitesimal extended point transformation in the phase space $(T^*Q, d\omega_Q)$. It will be referred to as the canonical lift of X. $\delta_V: V^{**} \to V$, or, in short δ, denotes the canonical isomorphism of the second dual of a linear space V onto V itself: $\langle f, \delta\,.\,F \rangle = \langle F, f \rangle$.

2. *General Outline of Mechanics of Local Group Spaces*

The general formulation of mechanics on a manifold can be simplified when the configuration space of a system is endowed with a teleparallelism. This simplification is especially remarkable when the constant vector-fields of the teleparallelism form a Lie algebra. The group manifolds of Lie groups is the most important example. They are used in the theory of such mechanical systems as material points in euclidean space, rigid bodies and affinely-rigid bodies. Some classical problems of mechanics on Lie groups have been investigated by Hermann (1972) and Arnold (1966).

Let us consider a mechanical system, the configuration space of which is a smooth manifold Q. Let V be some vector space.

Definition 2.1

A smooth mapping $\Omega : TQ \to V$ is said to endow Q with a teleparallelism if, for arbitrary $q \in Q$, the restriction $\Omega \mid T_qQ$ is a linear isomorphism of T_qQ onto V. The mapping Ω itself will be called a quasivelocity form.

Let $\rho : R \to Q$ be arbitrary motion and $\rho' : R \to TQ$ its natural lift to TQ (hodograph). $(\Omega \circ \rho')(t)$ is an Ω-quasivelocity of a system at time $t \in R$. In many problems $(\Omega \circ \rho')(t)$ is much more convenient than the usual generalised velocity $\rho'(t)$.

Ω is said to be holonomic if there exists an atlas $\{(U_\alpha, Q_\alpha)\}$ on Q such that Q_α take values in V and arbitrary motion satisfies the equation

$$\frac{d}{dt}(Q_\alpha \circ \rho) = \Omega_\alpha \circ \rho' \tag{2.1}$$

where $\Omega_\alpha = \Omega \mid TU_\alpha$. In general, Ω is non-holonomic. Ω gives rise to some V-valued differential form on Q, the so-called fundamental form of the teleparallelism, $\theta(\Omega)$:

$$\theta(\Omega)_q = \Omega \mid T_qQ \tag{2.2}$$

The abbreviation θ will be used as well.

Definition 2.2

An Ω-quasimomentum form is a mapping $\Sigma_\Omega : T^*Q \to V^*$ (denoted in short by Σ) defined as follows:

$$\Sigma \mid T_q^*Q = (\Omega \mid T_qQ)^{*-1} = \widetilde{(\Omega \mid T_qQ)} \tag{2.3}$$

for arbitrary $q \in Q$.

Hence, the linear mappings $\Omega \mid T_qQ$, $\Sigma \mid T_q^*Q$ are mutually cogradient. One could start with Σ as well, and then define Ω as a secondary construction.

Tensor bundles over manifolds with teleparallelisms become trivial. The corresponding trivialisations of the tangent and cotangent bundles are denoted as $t_\Omega : TQ \to Q \times V$, $t_\Sigma : T^*Q \to Q \times V^*$ respectively Obviously $t_\Omega(v) = (\tau_Q(v), \Omega(v))$, $t_\Sigma(p) = (\tau_Q^*(p), \Sigma(p))$ for arbitrary $v \in TQ$, $p \in T^*Q$. Arbitrary geometric object t on V gives rise–via Ω (more precisely via isomorphisms $\Omega \mid T_qQ$)–to some field over Q. Such a field will be referred as Ω-constant or the Maurer–Cartan field, and denoted as ${}_\Omega t$. When vector-fields X, Y are Ω-constant then the fundamental form satisfies the following Maurer–Cartan equations:

$$\langle d\theta(\Omega), X \wedge Y \rangle = -\tfrac{1}{2}\langle \theta(\Omega), [X, Y] \rangle \tag{2.4}$$

(cf. Kobayashi & Nomizu (1963) and Sternberg (1964)).

Let X be an arbitrary differentiable vector-field on Q and $\bar{X}$ its canonical lift to T^*Q. The generator F_X of $\bar{X}$ satisfies the following equation:

$$F_X(p) = (\Sigma(p), (\Omega \circ X \circ \tau_Q^*)(p)) \tag{2.5}$$

for arbitrary $p \in T^*Q$, where (f, v) denotes the value of $f \in V^*$ on $v \in V$. The following short-hand notation will be used as well:

$$F_X = (\Sigma, \Omega \circ X \circ \tau_Q^*) \tag{2.5a}$$

Proposition 2.1

The generator $F[v]$ of the canonical lift $\overline{_{\Omega}v}$ of the Maurer–Cartan vector-field $_{\Omega}v$, is given as:

$$F[v](p) = (\Sigma(p), v) \tag{2.6}$$

for arbitrary $p \in T^*Q$. We will write, in short,

$$F[v] = (\Sigma, v) \tag{2.6a}$$

Teleparallelism is said to be closed when its Maurer–Cartan vector-fields form a Lie algebra in the sense of a commutator, i.e. $\langle \Omega, [X, Y] \rangle$ = constant provided $\langle \Omega, X \rangle, \langle \Omega, Y \rangle$ are constant.

In this paper we are dealing with closed teleparallelisms only. Some problems concerning the non-closed case were investigated by Hermann (1972). When a teleparallelism is closed the exterior algebra of Maurer–Cartan differential forms is closed under the exterior differentiation. This follows from the Maurer–Cartan equations.

The use of quasivelocities and quasimomenta is especially justified and advantageous just as in the case of closed teleparallelisms. Arbitrary closed teleparallelism gives rise to a Lie algebra structure in V. The corresponding Lie bracket of vectors $u, v \in V$ will be denoted by the same symbol, as a commutator $[u, v]$. It is uniquely defined by $[_{\Omega}u, _{\Omega}v] = _{\Omega}[u, v]$.

The quasivelocity form Ω is holonomic if and only if the corresponding Lie algebra is commutative.

Proposition 2.2

Let Ω be a closed teleparallelism on Q. Denoting the generators of hamiltonian vector-fields $\overline{_{\Omega}v}$ as $F[v]$, we have the following Poisson brackets:

$$\{F[u], F[v]\} = F[[u, v]] \tag{2.7}$$

$$\{F[u], f \circ \tau_Q^*\} = (_{\Omega}u \,.\, f) \circ \tau_Q^* \tag{2.8}$$

$$\{f \circ \tau_Q^*, g \circ \tau_Q^*\} = 0 \tag{2.9}$$

for arbitrary $u, v \in V, f, g \in C^1(Q)$.

The equations above are sufficient to calculate any other Poisson bracket. In practical calculations one uses the coordinate form of (2.7), $\{F_A, F_B\} = \gamma_{AB}{}^C F_C$, where $\gamma_{AB}{}^C$ are structural constants with respect to some basis $\{e_A\}$, $[e_A, e_B] = \gamma_{AB}{}^C e_C$, and F_A is an abbreviation for $F[e_A]$.

Lagrangean mechanics becomes clearer when formulated in $Q \times V$ instead in TQ. Similarly, hamiltonian mechanics is more natural in $Q \times V^*$ than in T^*Q. Let us find the vector-fields on $Q \times V$ and $Q \times V^*$ corresponding to fields ${}_{\Omega}v'$, $\overline{{}_{\Omega}v}$ via the action of diffeomorphisms t_{Ω}, t_{Σ} respectively.

Proposition 2.3

Let Ωv, Σv be vector-fields on $Q \times V$, $Q \times V^*$ respectively, corresponding to ${}_{\Omega}v'$, $\overline{{}_{\Omega}v}$ via the canonical diffeomorphisms t_{Ω}, t_{Σ}. Then:

$$(\Omega v \,.\, F)(q, u) = \langle dF(\,.\,, u)_q, {}_{\Omega}v_q \rangle + (D_u F(q, .), [u, v]) \tag{2.10}$$

$$(\Sigma v \,.\, G)(q, f) = \langle dG(\,.\,, f)_q, {}_{\Omega}v_q \rangle + (f, [v, \delta \,.\, D_f G(q, .)]) \tag{2.11}$$

for arbitrary $q \in Q, u \in V, f \in V^*, F \in C^1(Q \times V), G \in C^1(Q \times V^*)$. (Symbols D_u, D_f denote the usual derivatives in the sense of differentiation on vector spaces.)

Instead of the usual phase space $(T^*Q, d\omega_Q)$ one often makes use of the phase space $(Q \times V^*, \gamma_{\Sigma})$, where $\gamma_{\Sigma} = t_{\Sigma}^{-1*} . d\omega_Q$. Let $\pi_1 : Q \times V^* \to Q$, π_2: $Q \times V^* \to V^*$ be natural projections onto the first and second component of the cartesian product respectively. Making use of the Poisson bracket $\{F, G\} = \langle dF \wedge dG, \tilde{\gamma}_{\Sigma} \rangle$ on $Q \times V^*$, one can rewrite Proposition 2.2 as follows:

$$\{f \circ \pi_1, g \circ \pi_1\} = 0 \tag{2.12}$$

$$\{(\delta^{-1} . v) \circ \pi_2, f \circ \pi_1\} = ({}_{\Omega}v . f) \circ \pi_1 \tag{2.13}$$

$$\{(\delta^{-1} . u) \circ \pi_2, (\delta^{-1} . v) \circ \pi_2\} = (\delta^{-1} . [u, v]) \circ \pi_2 \tag{2.14}$$

for arbitrary $u, v \in V, f, g \in C^1(Q)$. The equations above hold because

$$(\delta^{-1} . v) \circ \pi_2 \circ t_{\Sigma} = F[v]$$

Now let $L : TQ \to R$ be a Lagrangean of a system. Then, denoting the corresponding Legendre transformation as $\mathscr{L}: TQ \to T^*Q$ (cf. Abraham (1967)), we have for arbitrary $(q, \xi) \in Q \times V$:

$$(t_{\Sigma} \circ \mathscr{L} \circ t_{\Omega}^{-1})(q, \xi) = (q, D_{\xi}(L \circ t_{\Omega}^{-1})(q, .)) \tag{2.15}$$

Consequently one can simply use the Lagrangean Λ: $Q \times V \to R$ and define the Legendre transformation as a mapping $\mathfrak{L} : Q \times V \to Q \times V^*$ such that

$$\mathfrak{L}(q, \xi) = (q, D_{\xi} \Lambda)$$

In this paper we are mainly concerned with teleparallelisms induced by Lie groups of transformations. Let us consider a homogeneous space (Q, G, f), where Q is an analytic manifold, G a Lie group and $f: G \times Q \to Q$ an analytic mapping defining the action of G on Q. The abbreviation $f(g, q) = gq$ will be used in the case of left action (when $f(g_1 g_2, .) = f(g_1, f(g_2, .))$) and, similarly, $f(g, q) = qg$ for right actions (i.e. such as $f(g_1 g_2, .) = f(g_2, f(g_1, .))$). The Lie algebra of G will be denoted as $\mathfrak{g}$ and the Lie algebra of Killing fields of G on Q as $\mathfrak{g}^f$. The canonical isomorphism of linear spaces $\mathfrak{g}, \mathfrak{g}^f$ will be denoted as $\sigma_f : \mathfrak{g} \to \mathfrak{g}^f$. When f acts on the left then σ_f is an isomorphism of Lie algebras at the same time. When f acts on the right σ_f is a Lie algebra anti-isomorphism.

An almost group space is a homogeneous space of Lie group with discrete isotropy groups. It is called a group space simply when isotropy groups are trivial.

Almost group spaces are endowed with natural teleparallelisms because Killing vector-fields are then independent at any point. More strictly:

Arbitrary almost group space (Q, G, f) gives rise to the closed teleparallelism Ω^f: $TQ \to \mathfrak{g}$ such that

$$(\sigma_f \,.\, \Omega^f(v))_{\tau(v)} = v \tag{2.16}$$

for arbitrary $v \in TQ$. Roughly speaking, with arbitrary $v \in T_qQ$, there is associated the only Killing vector-field on Q, which equals v at q. In such a way the mechanics on almost group spaces becomes a special case of the general theory of systems with teleparallelisms. For example, Proposition 2.2 implies that

$$\{F[u], F[v]\} = F[u, v] \tag{2.17}$$

when G acts on the left; hence, $u \to F[u]$ is an isomorphism of Lie algebras. Similarly,

$$\{F[u], F[v]\} = F[v, u] \tag{2.18}$$

when G acts on the right; hence $u \to F[u]$ is an anti-isomorphism of Lie algebras.

Let t_f: $TQ \to Q \times \mathfrak{g}$, t^f: $T^*Q \to Q \times \mathfrak{g}^*$ be trivialisation mappings corresponding to the teleparallelism Ω^f. Now let Tg, T^*g denote the natural lifts of the action of $g \in G$ on Q to the manifolds TQ, T^*Q respectively. In $Q \times \mathfrak{g}$, $Q \times \mathfrak{g}^*$ they appear as follows:

$$t_f \circ Tg \circ t_f^{-1} = f(g, .) \times Ad_g \tag{2.19}$$

$$t^f \circ T^*g \circ t^{f\,-1} = f(g, .) \times Ad_g^{*-1} \tag{2.20}$$

when G acts on the left, and

$$t_f \circ Tg \circ t_f^{-1} = f(g, .) \times Ad_g^{-1} \tag{2.21}$$

$$t^f \circ T^*g \circ t^{f\,-1} = f(g, .) \times Ad_g^{*} \tag{2.22}$$

when G acts on the right.

The quantisation procedure for mechanical systems in manifolds has been formulated by Mackey (1963). According to him, probability amplitudes describing pure quantum states are given by complex densities of weight 1/2 on Q (cf. Sternberg (1964)). However, when Q is endowed with a teleparallelism, one can use scalar wave functions instead of such densities, because teleparallelisms give rise to natural measures. In fact, the Maurer–Cartan differential form ${}_{\Omega}\epsilon$, corresponding to arbitrary non-vanishing form ϵ of maximum degree on V, does not vanish anywhere and, consequently, it gives rise to a measure Δ on Q:

$$\int f(q)d\Delta(q) = \int f_{\Omega}\epsilon \tag{2.23}$$

It is unique up to a non-essential constant factor.

Hence, the theory can be formulated in Hilbert space $L^2(Q, \Delta)$ with the natural scalar product

$$\langle \psi_1 | \psi_2 \rangle = \int \bar{\psi}_1 \psi_2 d\Delta \tag{2.24}$$

'Position variables' are described by hermitean operators which multiply elements of $L^2(Q, \Delta)$ by measurable bounded functions C: $\hat{C}\Psi = C\Psi$. The second, very important class of physical quantities, is connected with infinitesimal mappings, i.e. vector-fields on Q. Let X be a vector-field on Q and $\rho[X]$ its 'divergence':

$$\underset{X}{\mathscr{L}} {}_{\Omega}\epsilon = \rho[X]\, {}_{\Omega}\epsilon \tag{2.25}$$

Transformation properties of 1/2-densities imply that the physical quantity corresponding to X should be described by the following operator (cf. Mackey (1963)):

$$\hat{F}[X] = \frac{\hbar}{i} X + \frac{\hbar}{2i} \rho[X] = \frac{\hbar}{i}(X + \tfrac{1}{2}\rho[X]) \tag{2.26}$$

Obviously, the natural domain of $\hat{F}[X]$ consists of differentiable functions in $L^2(Q, \Delta)$. The additional, non-differential term $(\hbar/2i)\rho[X]$ is due to the non-invariance of ${}_{\Omega}\epsilon$ and Δ with respect to X and assures $\hat{F}[X]$ to be formally self-adjoint,

$$\langle \Psi_1 | \hat{F}[X] \Psi_2 \rangle = \langle \hat{F}[X] \psi_1 | \Psi_2 \rangle \tag{2.27}$$

for arbitrary $\Psi_1, \Psi_2 \in C_0^\infty(Q)$.

Especially important are physical quantities corresponding to the Maurer-Cartan vector-fields. We will write $\hat{F}[v]$, $\rho[v]$ instead of $F[{}_{\Omega}v]$, $\rho[{}_{\Omega}v]$ respectively. Let us notice that for arbitrary $v \in V$, $\rho[v]$ is constant.

The quantum Poisson brackets of the position variables and Maurer-Cartan quantities are exactly the same as the classical ones (cf. Proposition 2.2).

Proposition 2.4

Let u, v be arbitrary vectors and C, D arbitrary smooth functions on Q. The following commutation rules are satisfied

$$\frac{1}{\hbar i}[\hat{F}[u], \hat{F}[v]] = \hat{F}[[u, v]] \tag{2.28}$$

$$\frac{1}{\hbar i}[\hat{F}[u], \hat{C}] = \widehat{{}_{\Omega}u \,.\, C} \tag{2.29}$$

$$\frac{1}{\hbar i}[\hat{C}, \hat{D}] = 0 \tag{2.30}$$

where all operators are assumed to act on $C_0^\infty(Q)$.

Now let (Q, G, f) be a homogeneous space. The left action of G on Q gives rise to some unitary representation of G in $L^2(Q)$, denotes as $g \mapsto W(g)$. Operators $W(g)$ are given by the Mackey formula:

$$W(g) = \sqrt{[D(g^{-1})} f(g^{-1}, .)^* \tag{2.31}$$

i.e.

$$(W(g)\Psi)(q) = [D(g^{-1})]^{1/2} \Psi(f(g^{-1}, q)) \tag{2.32}$$

where $D(g)$ is the 'determinant' of $f(g, .)$:

$$f(g^{-1}, .)^* {}_{\Omega}\epsilon = D(g) {}_{\Omega}\epsilon \tag{2.33}$$

(The n-form ${}_{\Omega}\epsilon$ describing the measure Δ is assumed to be positive with respect to the chosen orientation on Q.) When G acts on the right, $g \leftrightarrow W(g)$ is an anti-representation.

Obviously, $\hat{F}[v]$ are infinitesimal generators of W. If $\{g_t : t \in R\}$ is a one-parameter subgroup of G, generated by $v \in \mathfrak{g}$, then:

$$\frac{1}{\hbar i} \frac{d}{dt} (W(g_t)\Psi)_{t=0} = \hat{F}[v]\Psi$$

for arbitrary $\Psi \in C_0^\infty(Q)$.

The mechanical system becomes fully described when its dynamical structure, i.e. hamiltonian, is known. In physical problems which present interest, the hamiltonian, or at least its kinetic part, is built algebraically of quantities connected with the teleparallelism, i.e. of functions $F[u]$ on the classical level and operators $\hat{F}[u]$ in quantum theory. These quantities give rise to the local kinematical symmetries of a system (on the almost group spaces these symmetries become global). In the special case of interactions with maximal symmetry, hamiltonian is given by some Casimir invariant of a local group of kinematical symmetries.

Example 2.1a. The Mechanics in Linear Spaces

Let Q be an open subset of a vector space V. $\lambda_V : TV \to V \times V$, or, in short, λ, denotes the canonical diffeomorphism of TV onto $V \times V$ in the sense of a natural differential structure on V (cf. Abraham (1967)). It gives rise to the natural teleparallelism on Q denoted as $\Lambda_Q : TQ \to V$, or, briefly, Λ. Obviously

$$\Lambda = pr_2 \circ \lambda_V | TQ$$

where pr_i: $V \times V \to V, i = 1, 2$, is the canonical projection of the cartesian product onto its ith component. Obviously, $t_\Lambda = \lambda | TQ$. Generalised velocities and momenta are identified with elements of V, V^* respectively. Λ is closed and the corresponding Lie algebra is commutative. Hence Λ-velocities reduce to the usual holonomic velocities

$$\Lambda \circ \rho' = \frac{d\rho}{dt}$$

being the special case of (2.1) with $U_\alpha = V, Q_\alpha = id_V$.

Example 2.1b. The Mechanics in Affine Spaces

Holonomic teleparallelisms, with which we are dealing, in practical problems are connected with affine structures rather than with linear ones. Obviously the difference between these two cases is of a theoretical rather than a technical nature.

Let $(M, V, \to)$ be an affine space: M is a manifold, V-linear space of translations (free vectors) on M and '$\to$' is a mapping of $M \times M$ onto V satisfying the usual axioms (cf. Bourbaki (1955) and Ślebodziński (1970)). For arbitrary p, $q \in M, \overrightarrow{pq} \in V$ denotes the unique translation carrying p over into q.

Any $p \in M$ gives rise to a diffeomorphism $t_p : M \to V$ such that $t_p(q) = \overrightarrow{pq}$. Now let Q be an open subset of M. We have the following natural teleparallelism $A_Q : TQ \to V$ denoted briefly as A:

$$A_Q = \Lambda_V \circ Tt_p \,|\, TQ$$

This definition is correct because A_Q does not depend on the choice of p. Obviously, A_Q is holonomic.

Example 2.2. Non-Constrained Rigid Body

Let $(E, V, \to, g)$ be an euclidean space, i.e. $(E, V, \to)$ is an affine space and $g \in V^* \otimes V^*$ the metric tensor on the space of translations. Let $F(V, g)$ denote the manifold of g-orthonormal frames in V. The configuration space of a rigid body is given by the cartesian product $Q = E \times F(V, g)$, where E describes the translational (orbital) degrees of freedom and $F(V, g)$ gives an account of internal (spin-like) phenomena. Hence, the theory reduces to Example 2.1b and the example below.

Example 2.3. Rigid Body Fastened at One Point

Let us consider a rigid body without translational degrees of freedom, i.e. fastened at some point $p \in E$. Its configuration space $\{p\} \times F(V, g)$ identifies naturally with $F(V, g)$. There exist two natural homogeneous-space structures on the manifold $F(V, g)$. Let $O(V, g) \subset GL(V)$ denote the group of g-orthogonal mappings (i.e. linear isomorphisms of V preserving g) and $O(n) \subset GL(n)$, the group of real orthogonal matrices. Both groups act on $F(V, g)$ via mappings $l : O(V, g) \times F(V, g) \to F(V, g)$, $r : O(n) \times F(V, g) \to F(V, g)$:

$$l(A, \varphi) = (A\varphi_1, \ldots, A\varphi_n)$$

$$r(a, \varphi) = (\varphi_j a^j{}_1, \ldots, \varphi_j a^j{}_i, \ldots, \varphi_j a^j{}_n)$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)$. Obviously $(F(V, g), O(V, g), l)$ and $(F(V, g), O(n), r)$ are homogeneous spaces; l acts on the left and r on the right. These actions give rise to teleparallelisms:

$$\Omega^l : TF(V, g) \to \square(V, g),\ \Omega^r : TF(V, g) \to \square(n), \text{ where } \square(n)$$

is the space of skew-symmetric matrices and $\square(V, g)$ consists of g skew-symmetric linear mappings (i.e. $\langle g, \alpha v \otimes u \rangle = -\langle g, v \otimes \alpha u \rangle$ when $\alpha : V \to V$

belongs to $\square(V,g)$). Ω^l and Ω^r non-holonomic velocities are equal to the ordinary angular velocities referring to the laboratory and co-moving frame respectively. Similarly, Σ^l and Σ^r momenta describe the internal angular momentum (spin) in terms of laboratory and co-moving frame respectively.

3. *Configuration Space of an Affinely-Rigid Body*

Let us commence with some notions used in the modern theory of continua which, in the main, were elaborated by Toupin (1967).

Configurations of the medium are described by diffeomorphisms $\varphi : N \to M$, where M is a physical space and N the so-called material space. The configuration φ is to be understood in such a way that an X material point occupies the position $\varphi(X) \in M$. When particles are 'marked' by their initial positions then $N = M$.

Let $\mathrm{Dif}(N,M)$ denote the set of all diffeomorphisms of N onto M. Instead of $\mathrm{Dif}(N,N)$, $\mathrm{Dif}(M,M)$, abbreviations $\mathrm{Dif}N$, $\mathrm{Dif}M$ will be used. $\mathrm{Dif}(N,M)$ carries two natural homogeneous space structures with transformation groups $\mathrm{Dif}M$, $\mathrm{Dif}N$. These groups act on $\mathrm{Dif}(N,M)$ as follows:

$$\varphi \mapsto A \circ \varphi \tag{3.1}$$

$$\varphi \mapsto \varphi \circ B \tag{3.2}$$

According to (3.1) and (3.2), $\mathrm{Dif}M$ acts on the left and $\mathrm{Dif}N$ on the right. Obviously, groups $\mathrm{Dif}M$, $\mathrm{Dif}N$ do commute when acting on $\mathrm{Dif}(N,M)$. Symmetries of physical space and the corresponding conservation laws are formulated in terms of (3.1); similarly, transformations (3.2) are used when describing symmetries of the medium (cf. Toupin (1967) and Rogula (1966)).

$\mathrm{Dif}(N,M)$ is an infinite-dimensional configuration space of the medium.

The theory of an affinely-rigid body presupposes affine space-structures in both N and M. Let $(N, U, \to)$, $(M, V, \to)$ denote the corresponding affine spaces (cf. Example 2.1b). Using the same symbol '$\to$' to denote vectors in different affine spaces does not lead to any misunderstanding consequently, in what follows, this abbreviation will be used. Let us now fix some additional notations of affine objects.

Affine spaces $(U, U, \to)$, $(V, V, \to)$ are understood in the usual sense of affine geometry on vector spaces: $\overrightarrow{xy} = y - x$. $Af(N,M)$ denotes the set of affine mappings of N into M. $L(\xi) \in L(U, V)$ denotes a linear mapping of U into V, corresponding to $\xi \in Af(N,M)$: $\overrightarrow{\xi(p)\xi(q)} = L(\xi)\overrightarrow{pq}$. Obviously, $L : Af(N,M) \to L(U, V)$ is an epimorphism.

$Af(N, V)$ is a linear space with respect to the natural linear operations on mappings which take values in the vector space V. $Af(N,M)$ carries a natural affine structure, its space of translation is just $Af(N, V)$. When $F, G \in Af(N,M)$, then $\overrightarrow{FG} \in Af(N, V)$ is defined by:

$$\overrightarrow{FG}(p) = \overrightarrow{F(p)G(p)}$$

for arbitrary $p \in N$.

Configuration space of an affinely-rigid body could be defined as $Z = Af\,I(N,M)$, the set of affine isomorphisms of N onto M. Z is open in $Af(N,M)$. Consequently our problem reduces to Example 2.1b. The groups of affine isomorphisms of N and M act on Z according to (3.2) and (3.1).

Having practical applications in view, we will start with a slightly modified model of the configuration space. Let us distinguish some material point $O \in N$, e.g. the centre of the mass. It gives rise to the affine isomorphism $Y_O : Af(N,M) \to M \times L(U, V)$, where $L(U, V)$ is the space of linear mappings of U into V. For arbitrary $F \in Af(N,M)$,

$$Y_O(F) = (F(O), L(F))$$

Y_O maps the open subset $Z = Af\,I(N,M)$ onto $Q = M \times LI(U, V)$, where $LI(U, V)$ is the manifold of linear isomorphisms of U onto V. Finally:

The configuration space of an affinely-rigid body is defined as a manifold $Q = M \times LI(U, V)$. Therefore our problem reduces to Examples 2.1b (what concerns M) and 2.1a ($LI(U, V)$).

The factor M in Q describes the orbital degrees of freedom, i.e. the motion of distinguished particle $O \in N$. $LI(U, V)$ gives an account of internal degrees of freedom, i.e. affine rotations around the fixed particle O. When $U = R^n$, then $LI(U, V)$ identifies naturally with $F(V)$, the set of linear frames, and Q with $M \times F(V)$. However, the formulation in terms of abstract linear spaces and manifolds $LI(U, V)$ is more convenient and geometric.

Let us now investigate the internal degrees of freedom, i.e. the configuration space $W = LI(U, V)$. Hence we impose the constraints which forbid the point $O \in N$ to move in M. W carries three natural teleparallelisms:

(i) W is open in the vector space $L(U, V)$. Hence, it is endowed with the closed, integrable teleparallelism $\Lambda_W : TW \to L(U, V)$, denoted in short, by Λ (cf. Example 2.1a). $t_\Lambda : TW \to W \times L(U, V)$ denotes the corresponding natural identification.

(ii) The linear group in V, $GL(V)$ acts on $W = LI(U, V)$, according to (3.1), $\varphi \mapsto A\varphi = A \circ \varphi$. When endowed with this action W becomes a homogeneous space with trivial isotropy groups (a group space). This structure gives rise to the non-holonomic teleparallelism

$$\Omega_l : TW \to L(V)$$

where $L(V)$, the space of linear endomorphisms, is identified in a natural way with $\mathfrak{gl}(V)$, the Lie algebra of $GL(V)$.

(iii) $GL(U)$ acts on W according to (3.2): $\varphi \mapsto \varphi B = \varphi \circ B$. The corresponding teleparallelism will be denoted as

$$\Omega_r : TW \to L(U)$$

Teleparallelisms Ω_l, Ω_r become very simple when expressed in terms of Λ. In fact let us put $\omega_1 = \Omega_l \circ t_\Lambda^{-1}$, $\omega_r = \Omega_r \circ t_\Lambda^{-1}$. Then

$$\omega_1(\varphi, \xi) = \xi \circ \varphi^{-1}, \qquad \omega_r(\varphi, \xi) = \varphi^{-1} \circ \xi$$

(cf. Sławianowski (1974)).

Λ gives rise to the integration: $f \mapsto \int f(\varphi) d\theta(\varphi)$, which coincides obviously with the invariant Lebesgue integration on the vector space $L(U, V)$.

Both the teleparallelisms Ω_l, Ω_r lead to the same measure Δ on W. This is due to the unimodularity of linear groups (cf. Maurin (1968)). Obviously

$$\frac{d\Delta(\varphi)}{d\theta(\varphi)} = |\varphi|^{-n}$$

where $|\varphi|$ denotes the determinant of the matrix of φ with respect to some linear bases (the changing of bases results in the changing of normalisations only).

The Lebesgue measure is invariant under translations in $L(U, V)$ but not under the action of linear groups on $GL(U)$, $GL(V)$ on $L(U, V)$. Conversely Δ is $GL(U)$ and $GL(V)$ invariant, being at the same time non-invariant with respect to the local action of the abelian group $L(U, V)$ on W.

Now let us write down the Maurer–Cartan vector-fields of the teleparallelisms Λ, Ω_l, Ω_r. Let $a \in L(U, V)$, $\alpha \in L(V)$, $\beta \in L(U)$. Then

$$({}_\Lambda a \,.\, F)(\varphi) = (D_\varphi F, a)$$

$$(\alpha_l \,.\, F)(\varphi) = (D_\varphi F, \alpha \circ \varphi) \qquad (\beta_r \,.\, F)(\varphi) = (D_\varphi F, \varphi \circ \beta)$$

When the matrix elements of the mappings with respect to some bases $\{e_i\} \subset V$, $\{E_A\} \subset U$ are used as coordinates on $L(U)$, $L(V)$, $L(U, V)$, then

$${}_\Lambda a = a^i{}_A \frac{\partial}{\partial \varphi^i{}_A}$$

$${}_l\alpha = \alpha^i{}_j \varphi^j{}_A \frac{\partial}{\partial \varphi^i{}_A} \qquad {}_r\beta = \varphi^i{}_B \beta^B{}_A \frac{\partial}{\partial \varphi^i{}_A}$$

4. *Classical and Quantum Kinematical Symmetries*

Let us start with internal degrees of freedom. Quasivelocities Ω_l, Ω_r are non-holonomic because the linear groups are non-abelian. In what follows Ω_l, Ω_r will be called laboratory and co-moving quasivelocity forms respectively. The reason is that $\omega_l(\varphi, \xi)$ is a geometric object in the physical space M and $\omega_r(\varphi, \xi)$ in the material space N. Roughly speaking, Ω_l describes the behaviour of the body in terms of the laboratory system of reference and Ω_r in terms of the co-moving frame. They are related to each other by means of the configuration-mapping

$$\omega_r(\varphi, \xi) = \varphi^{-1} \circ \omega_l(\varphi, \xi) \circ \varphi \tag{4.1}$$

Ω_l, Ω_r provide us with an affine generalisation of the well-known laboratory and co-moving angular velocities of the usual metrically rigid body. They possess the same physical interpretation in terms of the eulerian and co-moving velocity fields (cf. Sławianowski (1974)),

$$V_{(\varphi,\xi)}(x) = \omega_l(\varphi, \xi) \,.\, x \qquad v_{(\varphi,\xi)}(X) = \omega_r(\varphi, \xi) \,.\, X \tag{4.2}$$

According to Section 2, quasimomentum forms corresponding to Λ, Ω_l, Ω_r should take values in $L(U, V)^*$, $L(V)^*$, $L(U)^*$ respectively. However, the spaces mentioned can be canonically identified with $L(V, U)$, $L(V)$, $L(U)$, respectively, via the formula:

$$(f, g) = \mathrm{Tr}\,(f . g) \tag{4.3}$$

Hence, as in the previous paper (Sławianowski, 1974), quasimomentum forms will be defined as mappings: $\Sigma : T^*W \to L(V, U)$, $\Sigma_l : T^*W \to L(V)$, $\Sigma_r : T^*W \to L(U)$. Let us put $\sigma_l = \Sigma_l \circ t_\Sigma^{-1}$, $\sigma_r = \Sigma_r \circ t_\Sigma^{-1}$. Then

$$\sigma_l(\varphi, \pi) = \varphi \circ \pi \qquad \sigma_r(\varphi, \pi) = \pi \circ \varphi \tag{4.4}$$

In what follows, all geometric objects describing teleparallelisms Ω_l, Ω_r will be related to $W \times L(U, V)$, $W \times L(V, U)$ rather than to TW, T^*W, respectively. This is achieved via identifications $t_\Lambda : TW \to W \times L(U, V)$, $t_\Sigma : T^*W \to W \times L(V, U)$. For example, the action of groups $GL(V)$, $GL(U)$ and the local action of $L(U, V)$ then take the following form:

$$(\varphi, \xi) \mapsto (A \circ \varphi, A \circ \xi), \qquad (\varphi, \xi) \mapsto (\varphi \circ B, \xi \circ B), \qquad (\varphi, \xi) \mapsto (\varphi + a, \xi) \tag{4.5}$$

$$(\varphi, \pi) \mapsto (A \circ \varphi, \pi \circ A^{-1}), \qquad (\varphi, \pi) \mapsto (\varphi \circ B, B^{-1} \circ \pi), \qquad (\varphi, \pi) \mapsto (\varphi + a, \pi) \tag{4.6}$$

The last three formulae describe the extended point transformations in the phase space (T, γ), where $T = W \times L(V, U)$ and $d\omega_W = t_\Sigma^* . \gamma$. Infinitesimal generators of these transformations, according to the formulae (2.5) and (2.6), take the following form:

$$F_l[\alpha](\varphi, \pi) = (\sigma_l(\varphi, \pi), \alpha) = \mathrm{Tr}\,(\varphi \circ \pi \circ \alpha) = \varphi^i{}_A \pi^A{}_j \alpha^j{}_i \tag{4.7}$$

$$F_r[\beta](\varphi, \pi) = (\sigma_r(\varphi, \pi), \beta) = \mathrm{Tr}\,(\pi \circ \varphi \circ \beta) = \pi^A{}_i \varphi^i{}_B \beta^B{}_A \tag{4.8}$$

$$F[a](\varphi, \pi) = (\pi, a) = \mathrm{Tr}\,(\pi \circ a) = \pi^A{}_i a^i{}_A \tag{4.9}$$

Obviously, the assignments $\alpha \mapsto F_l[\alpha]$, $\beta \mapsto F_r[\beta]$ are, respectively, representation and anti-representation of the commutator-Lie algebras $L(V)$, $L(U)$ into the Poisson bracket-Lie algebra over the phase space (T, γ). Therefore $\{F_l[\alpha], F_r[\beta]\} = 0$.

Physical quantities $F_l[\alpha]$, $F_r[\beta]$ will be called internal affine momenta, laboratory and co-moving respectively. They provide us with an affine generalisation of internal angular momenta (spins) of a metrically rigid body. Transformation from the laboratory to the co-moving frame is as follows:

$$F_r[\beta](\varphi, \pi) = F_l[\varphi \circ \beta \circ \varphi^{-1}](\varphi, \pi) \tag{4.10}$$

Now let us turn to the theory with translational degrees of freedom, i.e. to the configuration space $Q = M \times LI(U, V)$.

$\Phi : TQ \to V \times L(U, V)$ and $\mathfrak{R} : T^*Q \to V^* \times L(V, U)$ denote the natural teleparallelism and quasimomentum form respectively. The corresponding identification mappings are denoted as $t_\Phi : TQ \to Q \times V \times L(U, V)$, $t_{\mathfrak{R}} : T^*Q \to Q \times V^* \times L(V, U)$.

Among all transformation groups acting naturally on Q, especially important are $AfI(M)$, $GL(U)$. They act on Q as follows:

$$(m,\varphi)\xrightarrow[A]{}(A(m),L(A)\circ\varphi) \tag{4.11}$$

$$(m,\varphi)\xrightarrow[B]{}(m,\varphi\circ B) \tag{4.12}$$

The action (4.11) of $AfI(M)$ on Q gives rise to the homogeneous space structure. Obviously, for (4.12) and $GL(U)$ it is not the case. On the state spaces $Q \times V \times L(U,V)$, $Q \times V^* \times L(V,U)$, the groups mentioned act as follows:

$$(m,\varphi;\zeta,\xi)\xrightarrow[A]{}(A(m),L(A)\circ\varphi;L(A)\circ\zeta,L(A)\circ\xi) \tag{4.13}$$

$$(m,\varphi;\zeta,\xi)\xrightarrow[B]{}(m,\varphi\circ B;\zeta,\xi\circ B) \tag{4.14}$$

$$(m,\varphi;p,\pi)\xrightarrow[A]{}(A(m),L(A)\circ\varphi;p\circ L(A)^{-1},\pi\circ L(A)^{-1}) \tag{4.15}$$

$$(m,\varphi;p,\pi)\xrightarrow[B]{}(m,\varphi\circ B;p,B^{-1}\circ\pi) \tag{4.16}$$

Lie algebras of $AfI(M)$, $GL(U)$, will be identified with $Af(M,V)$, $L(U)$ respectively. Let $\mathscr{A}\in Af(M,V)$, $\beta\in L(U)$. The corresponding infinitesimal generators $F[\mathscr{A}]$, $\mathscr{K}[\beta]$ of extended point transformations (4.15) and (4.16) on the phase space $(Q\times V^*\times L(V,U), t_{\mathfrak{R}}^{-1*}.d\omega_Q)$ are given by:

$$F[\mathscr{A}](m,\varphi;p,\pi)=(p,\mathscr{A}(m))+(\varphi\circ\pi,L(\mathscr{A}))$$
$$=(p,\mathscr{A}(m))+\mathrm{Tr}(\varphi\circ\pi\circ L(\mathscr{A}))=(p,\mathscr{A}(m))+F_l[L(\mathscr{A})](\varphi,\pi) \tag{4.17}$$

$$\mathscr{K}[\beta](m,\varphi;p,\pi)=\mathrm{Tr}(\pi\circ\varphi\circ\beta)=F_r[\varphi,\pi] \qquad (4.18$$

Now let us fix some point $o\in M$. Arbitrary $\mathscr{A}\in Af(M,V)$ is uniquely given by $\alpha=L[\mathscr{A}]$, $a=\mathscr{A}(o)$. We then have the following decomposition:

$$F[\mathscr{A}]=P[a]+K_{\mathrm{orb}}[\alpha]+K_{\mathrm{int}}[\alpha]=P[a]+K[\alpha] \tag{4.19}$$

where

$$P[a](m,\varphi;p,\pi)=(p,a) \tag{4.20}$$

$$K_{\mathrm{int}}[\alpha](m,\varphi;p,\pi)=\mathrm{Tr}(\varphi\circ\pi\circ\alpha)=F_l[\alpha](\varphi,\pi) \tag{4.21}$$

$$K_{\mathrm{orb}}[\alpha](m,\varphi;p,\pi)=(\overrightarrow{om}\otimes p,\alpha)=\mathrm{Tr}((\overrightarrow{om}\otimes p)\circ\alpha) \tag{4.22}$$

In formula (4.22) the tensor $\overrightarrow{om}\otimes p\in V\otimes V^*$ is to be understood as identified with a linear function on $L(V)$ and, consequently, with some element of $L(V)$, cf. (4.3).

Obviously, $P[a]$ is an ath component of a linear momentum. $K_{\mathrm{orb}}[\alpha]$ will be referred as an αth component of the orbital affine momentum of the body with respect to the origin $o\in M$. Similarly, $K_{\mathrm{int}}[\alpha]$ describes the internal affine momentum, or an affine spin of the body with respect to the fixed material point $O\in N$. $K_o[\alpha]=K_{\mathrm{orb}}[\alpha]+K_{\mathrm{int}}[\alpha]$ is an αth component of the total affine momentum with respect to $o \in M$. Similarly, $\mathscr{K}[\beta]$ is a βth component of the comoving affine spin.

Let $Q^i, Q^i{}_A, P_i, P^A{}_i$ be canonical affine coordinates on $Q \times V^* \times L(V, U)$ corresponding to some bases $\{e_i\} \subset V$, $\{E_A\} \subset U$:

$$\overrightarrow{om} = Q^i(m, \varphi; p, \pi)e_i, \qquad p = P_i(m, \varphi; p, \pi)e^i$$
$$\varphi E_A = e_i Q^i{}_A(m, \varphi; p, \pi), \qquad \pi e_i = E_A P^A{}_i(m, \varphi; p, \pi)$$

Obviously,

$$t_{\mathfrak{R}}^{-1*} . d\omega_Q = dp_i \wedge dq^i + dP^A{}_i \wedge dQ^i{}_A$$

and

$$K_{\text{orb}}[\alpha] = K_{\text{orb}}{}^i{}_j \alpha^j{}_i = Q^i P_j \alpha^j{}_i \tag{4.23}$$
$$K_{\text{int}}[\alpha] = K_{\text{int}}{}^i{}_j \alpha^j{}_i = Q^i{}_A P^A{}_j \alpha^j{}_i \tag{4.24}$$
$$K_o{}^i{}_j = K_{\text{orb}}{}^i{}_j + K_{\text{int}}{}^i{}_j = Q^i P_j + Q^i{}_A P^A{}_j \tag{4.25}$$
$$\mathscr{K}[\beta] = \mathscr{K}^A{}_B \beta^B{}_A = P^A{}_i Q^i{}_B \beta^B{}_A \tag{4.26}$$

Now let us turn again to quantum mechanics. As mentioned above, the manifold $LI(U, V)$ is endowed with two natural measures, Δ and the Lebesgue measure θ of the vector space $L(U, V)$. Similarly, M is endowed with the V-invariant Lebesgue measure τ. Hence we have at our disposal two distinguished measures $\tau \otimes \theta$, $\tau \otimes \Delta$ on the configuration space $Q = M \times LI(U, V)$. Quantum theory can be formulated in $L^2(Q, \tau \otimes \theta)$ or in $L^2(Q, \tau \otimes \Delta)$. In practical problems, $L^2(Q, \tau \otimes \theta)$ is more convenient.

Let ∇_m denote a derivative at $m \in M$ in the usual sense of differentiating on affine spaces: for arbitrary $n \in M, D_{\overrightarrow{nm}}(\varphi \circ t_n^{-1}) = \nabla_m \varphi \in V^*$, where $t_n(q) = \overrightarrow{nq}$. Now according to (2.26) operators corresponding to physical quantities $P[a], K_{\text{int}}[\alpha], K_{\text{orb}}[\alpha]$ can be written as follows:

$$(\hat{P}[a]\Psi)(m, \varphi) = \frac{\hbar}{i}(\nabla_m \Psi(\,.\,, \varphi), a) \tag{4.27}$$

$$(K_{\text{int}}[\alpha]\Psi)(m, \varphi) = \frac{\hbar}{i}(D_\varphi \Psi(m, .), \alpha \circ \varphi) + \frac{hn}{2i}\operatorname{Tr}\alpha \tag{4.28}$$

$$(\hat{K}_{\text{orb}}[\alpha]\Psi)(m, \varphi) = \frac{\hbar}{i}(\nabla_m \Psi(\,.\,, \varphi), \alpha . \overrightarrow{om}) + \frac{\hbar}{2i}\operatorname{Tr}\alpha \tag{4.29}$$

where $n = \dim M$. Obviously:

$$\hat{K}_o[\alpha] = \hat{K}_{\text{orb}}[\alpha] + \hat{K}_{\text{int}}[\alpha] \tag{4.30}$$

Making use of affine coordinates $(x^i, \varphi^i{}_A)$ on Q, we have

$$\hat{P}[a] = \frac{\hbar}{i} a^i \frac{\partial}{\partial x^i} = a^i \hat{P}_i \tag{4.31}$$

$$\hat{K}_{\text{orb}}[\alpha] = \frac{\hbar}{i} \alpha^j{}_i x^i \frac{\partial}{\partial x^j} + \frac{\hbar}{2i} \alpha^i{}_i = \alpha^j{}_i \hat{K}_{\text{orb}}{}^i{}_j \tag{4.32}$$

where

$$\hat{K}_{\text{orb}}{}^{i}{}_{j} = \frac{\hbar}{i} x^{i} \frac{\partial}{\partial x^{j}} + \frac{\hbar}{2i} \delta^{i}{}_{j} \tag{4.33}$$

$$\hat{K}_{\text{int}}[\alpha] = \frac{\hbar}{i} \alpha^{j}{}_{i} \varphi^{i}{}_{A} \frac{\partial}{\partial \varphi^{j}{}_{A}} + \frac{\hbar n}{2i} \alpha^{i}{}_{i} = \alpha^{j}{}_{i} \hat{K}_{\text{int}}{}^{i}{}_{j} \tag{4.34}$$

where

$$\hat{K}_{\text{int}}{}^{i}{}_{j} = \frac{\hbar}{i} \varphi^{i}{}_{A} \frac{\partial}{\partial \varphi^{j}{}_{A}} + \frac{hn}{2i} \delta^{i}{}_{j} \tag{4.35}$$

Similarly, the co-moving affine momentum is given as follows:

$$(\hat{\mathscr{K}}[\beta]\Psi)(m, \varphi) = \frac{\hbar}{i} (D_{\varphi}\Psi(m, .), \varphi \circ \beta) + \frac{\hbar n}{2i} \operatorname{Tr} \beta \tag{4.36}$$

In affine coordinates

$$\hat{\mathscr{K}}[\beta] = \beta^{B}{}_{A} \hat{\mathscr{K}}^{A}{}_{B} \tag{4.37}$$

where

$$\hat{\mathscr{K}}^{A}{}_{B} = \frac{\hbar}{i} \varphi^{i}{}_{B} \frac{\partial}{\partial \varphi^{i}{}_{A}} + \frac{\hbar n}{2i} \delta^{A}{}_{B} \tag{4.38}$$

If we formulated the theory in $L^2(Q, \tau \otimes \Delta)$ there would not be any non-differential terms in operators $\hat{K}_{\text{orb}}[\alpha], \hat{K}_{\text{int}}[\alpha], \hat{K}_{o}[\alpha], \hat{\mathscr{K}}[\beta]$. However, such terms would then appear in $\hat{P}[a]$.

According to formulae (2.31), (2.32) and (2.33), finite actions of $AfI(M)$, $GL(U)$ on wave functions are given as follows:

$$(U(A)\Psi)(m, \varphi) = \det L(A)^{-(n+1)/2} \Psi(A^{-1}(m), L(A^{-1}) \circ \varphi) \tag{4.39}$$

$$(V(B)\Psi)(m, \varphi) = \det B^{-n/2} \Psi(m, \varphi \circ B^{-1}) \tag{4.40}$$

The additive abelian group $L(U, V)$ acts locally on wave functions with compact supports as follows:

$$(W(a)\Psi)(m, \varphi) = \Psi(m, \varphi - a) \tag{4.41}$$

5. *Euclidean Notions. Metrical Breaking of Affine Symmetry*

Up to now we have used only affine notions. It was quite sufficient when studing degrees of freedom and kinematics of an affinely-rigid body. However, in all practical problems, especially in dynamics, we are dealing with euclidean concepts. For example, the very definition of kinetic energy is based on the metric tensor in a physical space. This gives rise to the breaking of the aforementioned affine symmetry even before introducing interactions.

Let us turn the material and physical spaces N, M into euclidean spaces by endowing them with metric tensors $\eta \in U^* \otimes U^*, g \in V^* \otimes V^*$ respectively. In what follows, η will be referred to as the co-moving metric tensor and g as the physical metric tensor.

Let us take the manifold $EI(\eta, g) \subset LI(U, V)$, composed of isometries (euclidean isomorphisms) of the euclidean space (U, η) onto (V, g), when $\varphi \in EI(\eta, g)$, then $\eta = \varphi^* . g$. The manifold $\mathscr{G} = M \times EI(\eta, g) \subset Q$ is just the configuration space of the usual (i.e. metrical) rigid body. The groups of affine isometries $\mathscr{E}I(g) \subset Af\, I(M)$ and the orthogonal group $O(U, \eta) \subset GL(U)$ act on $\mathscr{G}$ according to (4.11) and (4.12). Obviously, such an approach becomes identical with that presented in examples (2.2) and (2.3) when putting $U = R^n$ and identify frames in V with mappings of R^n onto V.

When $\varphi \notin EI(\eta, g)$ we are dealing with a deformation. To describe deformations, one uses the whole menagerie of deformation tensors, e.g. The Euler tensor $e = \frac{1}{2}(\varphi^{-1*}\eta - g)$.

Let the configuration of an affinely-rigid body be $(m, \varphi) \in Q$ and its generalised velocity $(\zeta, \xi) \in V \times L(U, V)$. The metric tensors η, g enable us to define symmetric and skew-symmetric parts of quasivelocities $\omega_1(\varphi, \xi) = \xi \circ \varphi^{-1}$, $\omega_r(\varphi, \xi) = \varphi^{-1} \circ \xi$. Independent components of the skew-symmetric part $\omega_l^{[ij]} = \frac{1}{2}(\omega_l{}^i{}_k g^{kj} - \omega_l{}^j{}_k g^{ki})$ are just the laboratory components of angular velocity. Similarly, $\omega_r^{[AB]} = \frac{1}{2}(\omega_r{}^A{}_C \eta^{CB} - \omega_r{}^B{}_C \eta^{CA})$ are the co-moving components of angular velocity. Symmetric parts describe the deformative behaviour.

Let us now assume $\alpha \in L(V)$ to be g-skew-symmetric: $\langle g, \alpha v \otimes u \rangle = -\langle g, v \otimes \alpha u \rangle$. The generators $K[\alpha], K_{\text{int}}[\alpha], K_{\text{orb}}[\alpha]$ are laboratory components of angular momentum (the total angular momentum, spin and orbital angular momentum respectively). The same holds for the quantum counterparts $\hat{K}[\alpha], \hat{K}_{\text{int}}[\alpha], \hat{K}_{\text{orb}}[\alpha]$. When $\beta \in L(U)$ is η-skew-symmetric then $\mathscr{K}[\beta]$ (on the quantum level $- \hat{\mathscr{K}}[\beta]$) is a β-th component of the co-moving internal angular momentum.

We will now analyse the structure of the kinetic energy from the point of view of euclidean geometry in M. Let the positive, regular measure μ on the material space N describe a mass distribution in the body. Inertial properties of the body are described by its total mass, $M = \int_N d\mu$ (translational, i.e. orbital motion), and the co-moving tensor of inertia

$$J = \int \overrightarrow{OX} \otimes \overrightarrow{OX}\, d\mu(X) \in U \otimes U \tag{5.1}$$

(internal motion). O in formula (5.1) denotes the fixed point of the body (cf. Section 3). The total kinetic energy T equals the sum of the kinetic energies of all infinitesimal elements of the body. To give an account of the parametric dependence of T on g we will make use of the notation $g \leftrightarrow T_g$ when needed. Assuming O to coincide with the centre of mass,

$$\int_N \overrightarrow{OX}\, d\mu(X) = 0 \tag{5.2}$$

we have

$$T_g(m, \varphi; \zeta, \xi) = \tfrac{1}{2}\langle g, (\xi \otimes \xi) . J \rangle + \frac{M}{2}\langle g, \zeta \otimes \zeta \rangle$$
$$= \tfrac{1}{2} g_{ij} \xi^i{}_A \xi^j{}_B J^{AB} + \frac{M}{2} g_{ij} \zeta^i \zeta^j \tag{5.3}$$

Now let us make the natural assumption that $J \in U \otimes U$ is non-singular, and denote its reciprocal tensor as $\tilde{J} \in U^* \otimes U^*$. Let the lagrangean be of the form

$$L = T_g - V$$

where the potential V depends on the configuration only (no magnetic interactions). The corresponding Legendre transformation, $\mathscr{L}: Q \times V \times L(U, V) \rightarrow Q \times V^* \times L(V, U)$, is then reversible and the kinetic term of the hamiltonian $\mathscr{T}_g = T_g \circ \mathscr{L}^{-1}$ has the following form:

$$\mathscr{T}_g(m, \varphi; p, \pi) = \tfrac{1}{2} \langle \tilde{J}, (\pi \otimes \pi).\tilde{g} \rangle + \frac{1}{2M} \langle p \otimes p, \tilde{g} \rangle$$

$$= \tfrac{1}{2} \tilde{J}_{AB} \pi^A{}_i \pi^B{}_j g^{ij} + \frac{1}{2M} p_i p_j g^{ij} \qquad (5.4)$$

where $\tilde{g} \in V \otimes V$ is reciprocal to g. One can also write

$$\mathscr{T}_g = \tfrac{1}{2} J_{AB} P^A{}_i P^B{}_j g^{ij} + \frac{1}{2M} P_i P_j g^{ij} \qquad (5.5)$$

(where $P^A{}_i$, P_j are generalised momenta introduced in Section 4). The metric g breaks the affine symmetry $AfI(M)$. $\mathscr{T}_g$ is invariant with respect to the subgroup of isometries $\mathscr{E}I(g)$ only. Excepting angular momenta, the affine momenta fail to be constants of motion even when there are no interactions ($V = 0$). In fact

$$\{\mathscr{T}_g, \mathrm{K}[\alpha]\} = 2 \frac{\partial \mathscr{T}}{\partial g^{ij}} \alpha^{ij} \qquad (5.6)$$

The quantum operator of the kinetic energy is as follows:

$$\hat{T}_g = \tfrac{1}{2} \hat{J}_{AB} \hat{P}^A{}_i \hat{P}^B{}_j g^{ij} + \tfrac{1}{2} g^{ij} \hat{P}_i \hat{P}_j$$

$$= -\frac{\hbar^2}{2} \hat{J}_{AB} g^{ij} \frac{\partial^2}{\partial \varphi^i{}_A \, \partial \varphi^j{}_B} - \frac{\hbar^2}{2M} g^{ij} \frac{\partial^2}{\partial x^i \, \partial x^j} \qquad (5.7)$$

(cf. Section 4).

The symmetry properties of $\hat{T}_g$ are exactly the same as those of the classical quantity $\mathscr{T}_g$. Replacing in (5.6) all classical quantities by the corresponding operators and the classical Poisson bracket by the quantum one we obtain true equations.

Starting with the same degrees of freedom and kinematics of an affinely-rigid body we can construct another, quite alternative, dynamical model with affinely-invariant kinetic energy. Obviously, from the point of view of the usual mechanics, such a model is quite non-physical. However, it enables us to understand the structure of the physical model in more detail. It can also be useful when studying physical theories in amorphous space (cf. e.g. Bergmann & Brunnings, 1949). For example, it would be worthwhile reformulating the whole of mechanics and physics, on the grounds of amorphous affine geometry, without any metrical notions. It is interesting that in such theory the extended, or structured bodies, are more natural than the material points. In fact, when working with material points, we have no 'distance-like' scalar invariants which could be used as arguments in a potential energy of mutual interactions. On the other hand, when we consider extended, or structured bodies, such affine invariants do exist. Hence, it is above all mechanics in amorphous space which needs the notion of an affinely-rigid body. Also, an amorphous dynamical model of an affinely-rigid body is interesting from the point of view of pure mathematics and rational mechanics.

Both models—'physical' and 'amorphous'—become asymptotically equivalent in the case of infinitesimal deformations.

The main ideas of the 'amorphous' model are as follows: The physical space M is endowed with an affine geometry only (no fixed metric). In contrast, the material space N is assumed to be euclidean, i.e. endowed with the metric tensor $\eta \in U^* \otimes U^*$. The last assumption is quite natural; identifying configurations with affine frames (by means of main axes of inertia), we identify U with R^n at the same time. Hence, the natural Kronecker-metric in R^n induces some fixed metric on U.

Arbitrary configuration $(m, \varphi) \in Q$ of the body gives rise to the metric tensor $g(\varphi) = \varphi^{-1*}. \eta \in V^* \otimes V^*$ in the physical space M. All distances in M will now be measured by means of the configuration-dependent (and, consequently, matter-dependent) tensor $\varphi \mapsto g(\varphi)$. Putting $g(\varphi)$ instead of g into (5.3) we obtain just the 'amorphous kinetic energy' T_η:

$$T_\eta(m, \varphi; \zeta, \xi) = \tfrac{1}{2} \langle g(\varphi), (\xi \otimes \xi). J \rangle + \frac{M}{2} \langle g(\varphi), \zeta \otimes \zeta \rangle \tag{5.4}$$

One can easily show that

$$T_\eta(m, \varphi; \zeta, \xi) = \tfrac{1}{2} \langle \eta, (\omega_r(\varphi, \xi) \otimes \omega_r(\varphi, \xi))J \rangle + \frac{M}{2} \langle \eta, \lambda(\varphi, \zeta) \otimes \lambda(\varphi, \zeta) \rangle \tag{5.5}$$

where, according to formula (3.2), $\lambda(\varphi, \zeta) = \varphi^{-1}. \zeta$ is a translational quasi velocity corresponding to the action of the additive abelian group U on $Z = AfI(N, M)$. In our formulation, based on $Q = M \times LI(U, V)$, this action has the form:

$$(m, \varphi) \overset{}{\underset{u}{\longmapsto}} (r, \varphi), \qquad \text{where } \overrightarrow{mr} = \varphi . u \tag{5.6}$$

According to (3.2), quasivelocities $\lambda(\varphi, \zeta)$, $\omega_r(\varphi, \xi)$, taken together, form a co-moving affine quasivelocity corresponding to the right action of $AfI(N)$ on Z. Similarly, ζ, $\omega_l(\varphi, \xi)$ together form a laboratory affine quasivelocity connected with the left action of $AfI(M)$ on the configuration space. We did not mention (5.6) and λ in the previous sections because the translational quasivelocity is only needed when studying T_η. Roughly speaking, $\lambda(\varphi, \zeta)$ describes an orbital velocity in terms of the co-moving frame. Making use of affine coordinates we obviously have

$$T_\eta = \tfrac{1}{2}\,\eta_{AB}\Omega_r{}^A{}_C\Omega_r{}^B{}_D J^{CD} + \frac{M}{2}\eta_{AB}\lambda^A\lambda^B \tag{5.5a}$$

The most essential property of T_η is its invariance under the whole group $AfI(M)$ of laboratory affine symmetries.

Now let us assume again the Lagrangian to have a form $L = T_\eta - V$, where V is velocity-independent. The Legendre transformation $\mathscr{L}$ then leads to the following kinetic term $\mathscr{T}_\eta = T_\eta \circ \mathscr{L}^{-1}$ of the Hamiltonian,

$$\mathscr{T}_\eta = \tfrac{1}{2}\breve{J}_{AB}\mathscr{K}^A{}_C\mathscr{K}^B{}_D\eta^{CD} + \frac{1}{2M}\mathscr{P}_C\mathscr{P}_D\eta^{CD} \tag{5.7}$$

where $\mathscr{P}_A$ describes the co-moving orbital linear momenta p:

$$\mathscr{P}_A = p_i\varphi^i{}_A \tag{5.8}$$

Obviously, $\mathscr{P}_A$ are infinitesimal generators of (5.6). The aforementioned invariance of T_η under physical affine transformations is reflected by the following equations:

$$\{\mathscr{T}_\eta, K^i{}_j\} = \{\mathscr{T}_\eta, P_i\} = 0 \tag{5.9}$$

Quantum counterparts of (5.7) have a form

$$\hat{\mathscr{T}}_\eta = \tfrac{1}{2}\breve{J}_{AB}\hat{\mathscr{K}}^A{}_C\hat{\mathscr{K}}^B{}_D\eta^{CD} + \frac{1}{2M}\eta^{CD}\hat{\mathscr{P}}_C\hat{\mathscr{P}}_D \tag{5.10}$$

where

$$\hat{\mathscr{P}}_A = \varphi^i{}_A\hat{P}_i = \frac{\hbar}{i}\varphi^i{}_A\frac{\partial}{\partial x^i} \tag{5.11}$$

Invariance properties of (5.10) are exactly the same as those of (5.7):

$$[\hat{T}_\eta, K^i{}_j] = [\hat{T}_\eta, P_i] = 0$$

On the other hand, T_η still fails to be invariant under the right actions of $AfI(N)$ on Q. It is invariant only under the isometry subgroup $\mathscr{E}I(\eta) \subset AfI(N)$.

We conclude this section with a short philosophical remark. From the point of view of rationalistic *a priori*, the above 'non-physical' dynamical model based on the 'amorphous' kinetic energy T_η and configuration-dependent metric $g(\varphi)$, seems to be more consistent than the physical one. In fact, in the amorphous model, only dynamic interaction is able to break the affine, kinematical symmetry of degrees of freedom. The theory of dynamical systems on Lie groups (cf. Hermann (1972)) could profit greatly by the investigation of such a system.

6. *Towards Relativistic Theory*

Relativistic reformulation of the problem needs separate treatment. Here, we restrict ourselves to some general, guiding remarks.

We are dealing with two kinds of degrees of freedom: Orbital and internal ones (motion in M and in $L(U, V)$ respectively). It is possible to formulate the theory in which both kinds of motion are relativistic. However, it is hard to imagine constraints which would be able to keep the body affinely-rigid when its elements move relativistically with respect to the centre of mass. Besides, an infinite number of degrees of freedom is then involved because of the retardation and field-like mechanism of relativistic interactions.

It is much more easy to formulate a 'mixed' theory in which only orbital motion holds in a relativistic way. Retardation of the internal motion is neglected and all internal phenomena are referred to the proper time of the centre of mass. Such theory is able to describe small test bodies.

Many authors tried to include gyroscopic degrees of freedom into relativity along such lines (cf. Schild & Schlosser, 1965; Künzle, 1972). The approach we propose below is similar to those of Schild–Schlosser and Künzle. Obviously, we have more degrees of freedom (six deformative degrees in addition to three gyroscopic).

For the sake of simplicity we restrict ourselves to special relativity. Essentially, passing over to a curved space-time presents no difficulties. We remain on the classical level, for problems appear when one tries to quantise the theory.

Obviously the approximation neglecting the retardation in the internal motion does not break relativistic invariance. The theory is quite consistent and formulated completely in terms of Minkowskian geometry.

Let us briefly sketch the main ideas of the model. To derive the equations of motion we will make use of the homogeneous dynamics based on constraints in a symplectic manifold. We follow Dirac (1950, 1951, 1958), Tulczyjew (1968), Künzle (1969), Synge (1960) and Śniatycki & Tulczyjew (1971). They have formulated the only satisfactory method from the relativistic point of view.

Let $(X, \mathcal{U}, g)$ be a four-dimensional Minkowskian space, $(X, \mathcal{U})$ an affine space and $g \in \mathcal{U}^* \otimes \mathcal{U}^*$ a metric tensor with hyperbolic signature $(+---)$.

We will now construct an extended configuration space of the body (cf. Künzle, 1972), i.e. the space of all possible events (physical situations). This can be achieved in several equivalent ways. According to the ideas of Künzle (1972) and Schild & Schlosser (1965) we should make use of affine frames in X, consisting of one unit time-like vector and three space-like vectors orthogonal to the time-like one. However, we prefer to use the notions introduced in Section 3.

The material space is still assumed to be an euclidean space $(\mathcal{N}, U, \eta)$, where $\dim \mathcal{N} = 3$ and $\eta \in U^* \otimes U^*$ is a metric tensor. Hence, generalised events (situations) could be described by affine monomorphisms (injections) of $\mathcal{N}$ into X, with space-like images. However, it is much more convenient to divide the motion of the body onto the orbital motion of some fixed material particle $O \in \mathcal{N}$ (the centre of mass, for example) and the internal, relative motion with respect to O (cf. Section 3). Therefore, an extended configuration of an affinely-rigid body is a differential manifold

$$\mathcal{Y} = X \times LM_s(U, \mathcal{U})$$

where $LM_s(U, \mathcal{U})$ consists of all linear monomorphisms of U into $\mathcal{U}$, with space-like images ($\varphi^* . g$ is negatively defined, provided $\varphi \in LM_s(U, \mathcal{U})$). To return to the notation of Schild–Schlosser and Künzle, one should put $U = R^3$. The natural basis of R^3 is then mapped by $\varphi \in LM_s(R^3, \mathcal{U})$ onto a space-like triad in $\mathcal{U}$, which can be uniquely completed to a base in $\mathcal{U}$ (when the direction of time is fixed.)

A situation (event) $(x, \varphi) \in \mathcal{Y}$ is to be understood as follows: (i) $x \times X$ is a space-time location of the fixed point $O \in \mathcal{N}$, (ii) a material point $A \in \mathcal{N}$ is situated in space-time in such a way that its radius-vector with respect to x equals $\varphi . \overrightarrow{OA}$.

On the sixteen-dimensional manifold $\mathcal{Y}$, the linear group $GL(U)$ and the affine group $AfI(X)$ act naturally, according to the formulae:

$$(x, \varphi) \underset{A}{\longmapsto} (x, \varphi \circ A) \qquad A \in GL(U) \tag{6.1}$$

$$(x, \varphi) \underset{B}{\longmapsto} (B(x), L(B) \circ \varphi) \qquad B \in AfI(X) \tag{6.2}$$

Especially important are the following subgroups: η-orthogonal group $EI(\eta) \subset GL(U)$ and the Poincare group $\mathcal{E}I(g) \subset AfI(X)$. Finally, the abelian (additive) group $L(U, \mathcal{U})$ acts locally on $\mathcal{Y}$:

$$(x, \varphi) \mapsto (x, \varphi + \xi), \qquad \xi \in L(U, \mathcal{U}) \tag{6.3}$$

$\mathcal{Y}$ is an affine space, hence, instead of $T\mathcal{Y}$, $T^*\mathcal{Y}$, the manifolds $\mathcal{P}_L = \mathcal{Y} \times \mathcal{U} \times L(U, \mathcal{U})$, $\mathcal{P} = \mathcal{Y} \times \mathcal{U}^* \times L(\mathcal{U}, U)$ will be used. Obviously, $L(\mathcal{U}, U)$ will be identified with $L(U, \mathcal{U})^*$ via the formula $\langle f, g \rangle = \mathrm{tr}(f \circ g)$. Natural identification will be denoted as:

$$T_L : \mathcal{P}_L \to T\mathcal{Y}, \qquad T : \mathcal{P} \to T^*\mathcal{Y} \,. \mathrm{Dim}\, \mathcal{P}_L = \dim \mathcal{P} = 32$$

The symplectic manifold $(\mathcal{P}, T^* d\omega_{\mathcal{Y}})$ will be used as an extended phase space (super-phase space) of the problem.

Let $x^\mu, Q^\mu{}_A\ p_\mu, P^A{}_\mu$ be canonical coordinates on $\mathcal{P}$ corresponding to linear bases $\{E_A\} \subset U$, $\{e_i\} \subset \mathcal{U}$ and to some fixed origin $o \in X$ (cf. Section 4). Obviously,

$$T^* \omega_{\mathcal{Y}} = p_\mu\, dx^\mu + P^A{}_\mu\, dQ^\mu{}_A$$

Constraints corresponding to equations of motion will be introduced in two steps. We commence with the construction of the so-called kinematical constraints $\mathcal{M}$. They do not describe interactions, rather they define the proper physical degrees of freedom. In the theory of the relativistic gyroscope these constraints are achieved by imposing the orthogonality condition on the orbital 4-momentum and the triad of the space-like vectors of the tetrad. This is the only natural way to divide the history of the body onto equivalence classes of simultaneous events. Such a construction of kinematical constraints does not presuppose any metric structure in X. When the total electric charge of the body, or the external electromagnetic field, vanishes then

$$\mathcal{M} = \{(x, \varphi; p, \pi) : \varphi^* p = p \circ \varphi = 0\} \tag{6.4}$$

Obviously, such constraints are non-holonomic because they restrict the canonical momenta when positions are fixed. In coordinates

$$\mathcal{M} = \{z \in \mathcal{P} : \Phi_A(z) = 0\} \tag{6.5}$$

where

$$\Phi_A = p_\mu Q^\mu{}_A \tag{6.6}$$

Obviously, $\{\Phi_A, \Phi_B\} = 0$, hence $\mathcal{M}$ is what Dirac and others call 'first-class constraints' (cf. Dirac, 1958; Śniatycki, 1973). Therefore $\mathcal{M}$ is foliated by three-dimensional singular fibres (because codim $\mathcal{M} = 3$, dim $\mathcal{M} = 29$).

Interactions are described by imposing additional, non-holonomic constraints. All is based on the assumption that the energy of internal motion modifies the mass of the body. Let us introduce a function $\Phi : \mathcal{P} \to R$ given by

$$\Phi(x, \varphi; p, \pi) = \langle p \otimes p, \bar{g} \rangle - (m + H_i)^2 = p_\mu p^\mu - (m + H_i)^2 \tag{6.7}$$

H_i describes the energy of internal motion (related to the proper time of the centre of the mass) and $M = m + H_i$ is the total observed mass of the body (in such units that $c = 1$). The shape of H_i depends on the details of the dynamical model. For example, when there are no external fields and only elastic internal phenomena are involved, one can then expect

$$H_i = \tfrac{1}{2}\tilde{J}_{AB} P^A{}_\mu P^B{}_\nu \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + V(g_{\mu\nu} Q^\mu{}_A Q^\nu{}_B) \tag{6.8}$$

where the potential V depends on the Green deformation tensor and J is the co-moving tensor of inertia.

Dynamical constraints are given by

$$M = \{z \in \mathcal{P} : \Phi_A(z) = 0, \Phi(z) = 0\} \tag{6.9}$$

Obviously, $\dim M = 28$ ($\operatorname{codim} M = 4$). Dynamically allowed motions are described by curves tangent to the singular foliation $K(M)$ of the constraints M. If k is an arbitrary vector tangent to such a curve at $z \in M$, then $(k \lrcorner \gamma_z) | T_z M = 0$ (where $\gamma = T^* d\omega_{\mathcal{Y}}$). There exists non-physical arbitrariness of gauge—the motion starting at $z \in M$ is given by the whole fibre of $K(M)$ passing through z rather than by any one of the singular curves. In practical problems this arbitrariness disappears after projecting the curves to the extended configuration space $\mathcal{Y} = X \times LM_s(U, \mathcal{U})$. Dynamically allowed motions are then described by one-dimensional curves in $\mathcal{Y}$ and the only remaining arbitrariness is that of parametrisation of curves. (Usually, the proper time of the centre of mass is used as a parameter.) We did not assume external fields, hence, projecting motions to X, we obtain the straight lines. When H_i is given by (6.8), $\{\Phi, \Phi_A\} = 0$, hence M are first-class constraints and fibres of $K(M)$ are four-dimensional. The reduced phase space (quotient manifold of M with respect to $K(M)$ endowed with the natural symplectic structure) is twenty-four-dimensional, giving 12 degrees of freedom of an affinely-rigid body. This agrees with non-relativistic theory.

The constraints M need not be of I class and, in general, when interactions are present, they are not. Such situations were studied by Dirac (1958) in his generalised mechanics.

Let us finish with some remarks concerning an affinely-rigid body in an electromagnetic field. Let A be a vector-potential of this field. It seems reasonable to replace the functions Φ_B by $\Phi_B[A]$, where

$$\Phi_B[A] = (p_\mu - eA_\mu)\mathscr{Q}^\mu{}_B \tag{6.10}$$

i.e.

$$\mathscr{M} = \{(x, \varphi; p, \pi) : \varphi^* . (p - eA) = 0\} \tag{6.11}$$

Similarly, Φ should be replaced by

$$\Phi[A] = (p_\mu - eA_\mu)(p^\mu - eA^\mu) - (m + H_i[A])^2$$

where $H_i[A]$ is an internal hamiltonian describing the interaction of internal degrees of freedom with the electromagnetic field. $H_i[A]$ depends on the mechanical and electromagnetic structure of the body.

Instead of guessing constraints M describing interactions, one could try to follow Künzle (1972) and formulate the dynamics by means of the appropriate pre-symplectic structure on the velocity-space $\mathscr{P}_L$. Such an approach is more convenient in classical problems. However, when quantising the theory, the method based on constraints in $\mathscr{P}$ is more natural.

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